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Higher dimensional generalizations of the chiral field equations

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Abstract

We consider the self-dual Yang–Mills equation and its reduction, the Manakov–Zakharov system. We discuss three- and four-dimensional generalizations of the chiral field equations, and explain methods for constructing their exact solutions.

KEYWORDS

chiral field equations, dressing method, soliton solution

1 | INTRODUCTION

Two-dimensional integrable relativistic-invariant systems are classical objects of the theory of integrable systems (see Refs. 1–3). The first $(2+1)$ integrable system was proposed by Manakov and Zakharov in Ref. [6] as a reduction of the self-dual Yang–Mills (sdYM) equations (see Refs. 4, 5). The inverse spectral transform (IST) was first applied to the sdYM equations in Ref. [7] to construct local solutions. General solutions of the Manakov–Zakharov system, as well as localized soliton solutions, were constructed in Ref. [6].

The authors of Ref. [6] were certain that the Manakov–Zakharov system is relativistically invariant, but the system was only "semiinvariant," or relativistically invariant in the linear approximation. This fact was discovered by R.S. Ward[8] in 1988 who proposed another $(2+1)$ generalization of the chiral field equation. During the next 10 years, several other articles on this subject were published,[9–13] and then interest in this topic faded.

In this paper, we consider various reductions of the sdYM equations, discuss three- and four-dimensional generalizations of the chiral field equations, and explain methods for constructing their exact solutions.

2 | THE GENERAL MANAKOV–ZAKHAROV SYSTEM

Let Z_1, Z_2, Z_3, Z_4 be complex variables and let $Y(Z_1, \ldots, Z_4)$ be an $N \times N$ complex-valued invertible matrix. We consider the system

$$\frac{\partial}{\partial Z_1}\left(\frac{\partial}{\partial Z_4} Y \cdot Y^{-1}\right) - \frac{\partial}{\partial Z_3}\left(\frac{\partial}{\partial Z_2} Y \cdot Y^{-1}\right) = 0. \tag{1}$$

The sdYM for the gauge group SU(N) may be obtained from this system by the reduction $\bar{Z}_1 = Z_4 = z$ and $\bar{Z}_3 = -Z_2 = y$ (see Ref. 6).

The system (1) admits the following Lax representation:

$$\left(\lambda \frac{\partial}{\partial Z_1} + \frac{\partial}{\partial Z_2}\right)\Psi + A\Psi = 0,$$
$$\left(\lambda \frac{\partial}{\partial Z_3} + \frac{\partial}{\partial Z_4}\right)\Psi + B\Psi = 0. \tag{2}$$

Here, $\lambda \in \mathbb{C}$ is a spectral parameter. Indeed, assume that Ψ has the asymptotic expansion

$$\Psi \to I + \frac{P}{\lambda} + \cdots \quad \text{at } \lambda \to \infty$$

at infinity. Setting $\lambda = 0$, we obtain

$$A = -\frac{\partial Y}{\partial Z_2} Y^{-1}, \quad B = -\frac{\partial Y}{\partial Z_4} Y^{-1}, \tag{3}$$

where $Y = \Psi|_{\lambda=0}$. On the other hand, setting $\lambda \to \infty$, we find

$$A = -\frac{\partial P}{\partial Z_1}, \quad B = -\frac{\partial P}{\partial Z_3}, \quad \frac{\partial A}{\partial Z_3} = \frac{\partial B}{\partial Z_1}. \tag{4}$$

By combining (3) and (4), we obtain the system (1). Notice that in the special case when $Z_3 = -Z_4$, $Z_2 = Z_1$, the system (1) takes the symmetric form

$$\frac{\partial}{\partial Z_1}\left(\frac{\partial}{\partial Z_3} Y \cdot Y^{-1}\right) + \frac{\partial}{\partial Z_3}\left(\frac{\partial}{\partial Z_1} Y \cdot Y^{-1}\right) = 0. \tag{5}$$

Then, the Lax pair (2) takes the form

$$\frac{\partial \Psi}{\partial Z_1} + \frac{A}{\lambda + 1}\Psi = 0, \quad \frac{\partial \Psi}{\partial Z_3} + \frac{B}{\lambda - 1}\Psi = 0. \tag{6}$$

If Z_1, Z_3 are real, Equation (5) is relativistically invariant. In general, it is a complexification of the integrable equation for the chiral field studied in Refs. [1–3, 6].

The symmetric equation is a Lagrangian system with Lagrangian

$$L = \mathrm{tr}\left(\frac{\partial Y}{\partial Z_1} Y^{-1} \frac{\partial Y}{\partial Z_3} Y^{-1}\right). \tag{7}$$

We call Ψ and Ψ^{-1} the wave function and the inverse wave function. The simplest example of wave functions is

$$\Psi = I + \frac{\lambda_0 - \mu_0}{\lambda - \lambda_0} P, \quad \Psi^{-1} = I - \frac{\lambda_0 - \mu_0}{\lambda - \mu_0} P, \tag{8}$$

where $P = P^2$ is a projection operator, and $\lambda_0 \neq \mu_0$ are complex numbers. This case was considered in Ref. [15] in 1979. We consider only the simplest case when the rank of P is 1. In this case, P is a bivector

$$P = \frac{|p\rangle\langle q|}{\langle q|p\rangle}. \tag{9}$$

The system (2) can be rewritten as follows:

$$A = -\left(\lambda \frac{\partial}{\partial Z_1} + \frac{\partial}{\partial Z_2}\right)\Psi \cdot \Psi^{-1},$$
$$B = -\left(\lambda \frac{\partial}{\partial Z_3} + \frac{\partial}{\partial Z_4}\right)\Psi \cdot \Psi^{-1}. \tag{10}$$

According to (8) and (10), A and B are rational functions with simple poles at $\lambda = \lambda_0, \lambda = \mu_0$. We require the residues at those poles to be zero and rewrite the system (10) as follows:

$$A, B = \frac{\lambda_0 - \mu_0}{\lambda - \lambda_0} D_\lambda P\left(I - \frac{\lambda_0 - \mu_0}{\lambda - \mu_0} P\right). \tag{11}$$

Here, the symbol D_λ represents two operators (corresponding to A and B):

$$D_\lambda = \lambda \frac{\partial}{\partial Z_1} + \frac{\partial}{\partial Z_2}, \lambda \frac{\partial}{\partial Z_3} + \frac{\partial}{\partial Z_4}, \quad D_\lambda P = D_\lambda |p\rangle\langle q| + |p\rangle D_\lambda \langle q|.$$

Now let λ tend to λ_0, then

$$I - \frac{\lambda_0 - \mu_0}{\lambda - \mu_0} P \to I - P = \hat{P}, \tag{12}$$

where $\hat{P}$ is also a projection operator: $\hat{P}^2 = \hat{P}$. Moreover,

$$P\hat{P} = \hat{P}P = 0. \tag{13}$$

By virtue of (13), we obtain $\langle q|\hat{P}| = 0$. We henceforth assume that Ψ is a 2×2 matrix. This means that $\hat{P}$ is also a bivector, like P. Thus,

$$\hat{P} = \frac{|f\rangle\langle g|}{\langle g|f\rangle}. \tag{14}$$

By virtue of (13), $\langle g|p\rangle = 0$ and $\langle q|f\rangle = 0$.

Formally speaking, potentials A, B are rational functions with two poles at $\lambda = \lambda_0$ and $\lambda = \mu_0$. In fact, they do not depend on λ, so the residues at both poles must be zero. Canceling of the residue at $\lambda = \lambda_0$ leads to the equation

$$[D_{\lambda_0} P](1 - P) = 0. \tag{15}$$

Canceling of the residue at $\lambda = \mu_0$ gives

$$[-D_{\mu_0} P] \cdot P = [D_{\mu_0}(1 - P)] \cdot P = 0. \tag{16}$$

Here,

$$D_{\lambda_0} = \begin{cases} \lambda_0 \dfrac{\partial}{\partial Z_1} + \dfrac{\partial}{\partial Z_2} \\ \lambda_0 \dfrac{\partial}{\partial Z_3} + \dfrac{\partial}{\partial Z_4} \end{cases}, \quad D_{\mu_0} = \begin{cases} \mu_0 \dfrac{\partial}{\partial Z_1} + \dfrac{\partial}{\partial Z_2} \\ \mu_0 \dfrac{\partial}{\partial Z_3} + \dfrac{\partial}{\partial Z_4}. \end{cases} \tag{17}$$

Plugging (9) into (15) and (16), one finds

$$|D_{\mu_0} f\rangle = 0, \quad \langle D_{\lambda_0} q| = 0. \tag{18}$$

These equations can be resolved as follows. Let

$$\Psi_0 = T(Z_1 - \lambda Z_2, Z_3 - \lambda Z_4) \tag{19}$$

be a general solution to the system

$$\begin{gathered} \left(\lambda \frac{\partial}{\partial Z_1} + \frac{\partial}{\partial Z_2}\right)\Psi_0 = 0, \\ \left(\lambda \frac{\partial}{\partial Z_3} + \frac{\partial}{\partial Z_4}\right)\Psi_0 = 0. \end{gathered} \tag{20}$$

Let

$$F = T(Z_1 - \lambda_0 Z_2, Z_3 - \lambda_0 Z_4), \quad G = T^{-1}(Z_1 - \mu_0 Z_2, Z_3 - \mu_0 Z_4). \tag{21}$$

Then, Equation (18) can be resolved as follows:

$$|f\rangle = G\,|f_0\rangle, \quad \langle q| = \langle q_0|\,F. \tag{22}$$

Here, $|f_0\rangle$ and $\langle q_0|$ are a constant vector and covector, respectively. If the vector $|f\rangle$ and the covector $\langle q|$ are known, then the reconstruction of $|g\rangle$ and $\langle p|$ is a purely algebraic problem which we will discuss later. As long as P is known, the solution Y is

$$Y = 1 - \frac{\lambda_0 - \mu_0}{\mu_0} P, \quad Y^{-1} = 1 + \frac{\lambda_0 - \mu_0}{\mu_0} P. \tag{23}$$

The trick that we have used for the construction of the exact solution of the nonlinear system (1) is the simplest example of the powerful "dressing method," which was invented in 1974.[14] The above is an example of a dressing on the trivial background. To construct a more general class of solutions, we start with an arbitrarily chosen solution of the Lax system (2), which we denote $\Psi(\lambda, Z_1, Z_2, Z_3, Z_4)$. Then, as before

$$Y = \Psi(0, Z_1, Z_2, Z_3, Z_4). \tag{24}$$

Now we denote

$$F = \Psi(\lambda, Z_1, Z_2, Z_3, Z_4), \quad G = \Psi^{-1}(\mu, Z_1, Z_2, Z_3, Z_4). \tag{25}$$

We seek a new solution $\hat{\Psi}$ of the system (2) as follows:

$$\hat{\Psi} = \left(1 + \frac{\lambda_0 - \mu_0}{\lambda - \lambda_0}\hat{P}\right)\Psi, \quad \hat{\Psi}^{-1} = \Psi\left(1 - \frac{\lambda_0 - \mu_0}{\lambda - \mu_0}\hat{P}\right). \tag{26}$$

In (26), λ_0 and μ_0 are arbitrary complex numbers, and $P = P^2$, $\hat{P} = 1 - P$ are complimentary projectors. In the case $N = 2$, they are bivectors

$$\hat{P} = \frac{|\hat{p}\rangle\langle\hat{q}|}{\langle\hat{p}|\hat{q}\rangle}, \quad 1 - \hat{P} = \frac{|\hat{f}\rangle\langle\hat{g}|}{\langle\hat{f}|\hat{g}\rangle}. \tag{27}$$

The new vector $|\hat{f}\rangle$ and covector $\langle\hat{q}|$ are given by expressions

$$|\hat{f}\rangle = G|\hat{f}_0\rangle, \quad \langle\hat{q}| = \langle\hat{q}_0|F, \tag{28}$$

where F and G are given by expressions (25).

The new solution of the system (1) is given by expressions

$$\hat{Y} = \left(1 - \frac{\lambda_0 - \mu_0}{\lambda_0}\hat{P}\right)Y, \quad \hat{Y}^{-1} = Y^{-1}\left(1 + \frac{\lambda_0 - \mu_0}{\mu_0}\right)P. \tag{29}$$

The constructed solution is called a one-soliton solution, but this is not an exact term. We call it a "one-pole solution" instead. The construction of the one-pole solution can be generalized to produce the much more general "n-pole solution."

We again use the dressing method. Suppose we know one solution of (2) with wave function $\Psi(\lambda, Z_1 \dots Z_4)$. We seek the dressed n-pole solution in the form

$$\tilde{\Psi} = \chi(\lambda)\Psi(\lambda),$$

where $\chi(\lambda)$ is a rational function with simple poles. One can seek $\chi(\lambda)$ in the following form:

$$\chi = I + \sum_{k=1}^{n} \frac{R_k}{\lambda - \lambda_k}. \tag{30}$$

The inverse function is also rational:

$$\chi^{-1} = I + \sum_{k=1}^{n} \frac{S_k}{\lambda - \mu_k}. \tag{31}$$

Here, λ_k and μ_k are certain complex numbers.

We consider only the case when R_k and S_k have rank 1. In this case, they can be presented as tensor products

$$R_k = |p_k\rangle\langle q_k|, \quad S_k = |f_k\rangle\langle g_k|. \tag{32}$$

The vectors $|f_k\rangle$ and the covectors $\langle q_k|$ can be found by the use of the dressing function Ψ_0, obeying the following equations

$$|f_k\rangle = G_k|f_{0\,k}\rangle, \quad \langle q_k| = \langle q_{0\,k}|F_k, \quad F_k = \Psi_0^{-1}(\lambda_k), \quad G_k = \Psi_0(\mu_k). \tag{33}$$

Here, $|f_{0\,k}\rangle$ and $\langle q_{0\,k}|$ are arbitrary constant vectors and covectors. The remaining components of (32), namely, the vectors $|p_k\rangle$ and covectors $\langle g_k|$, can be found by solving the following systems of linear algebraic equations:

$$|f_l\rangle + \sum_{k=1}^{n} \frac{|p_k\rangle\langle q_k|f_l\rangle}{\mu_l - \lambda_k} = 0,$$

$$|q_l\rangle + \sum_{k=1}^{n} \frac{|g_k\rangle\langle f_k|q_l\rangle}{\lambda_l - \mu_k} = 0. \tag{34}$$

If all λ_l and μ_k are distinct, then the denominators in (34) are nonzero and this system has a unique solution. Special cases when some λ_k and μ_k coincide can be studied by taking a limit from the general case.

We note that expressions (30) and (31) do not change if the poles $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ are reordered. There is another way to construct n-pole solutions. Suppose we define some ordering of poles and seek the wave function as a product of one-pole solutions (see Ref. 15):

$$\chi = \prod_{k=1}^{n} \left(I + \frac{\lambda_k - \mu_k}{\lambda - \lambda_k} P_k \right), \quad \chi^{-1} = \prod_{k=1}^{n} \left(I - \frac{\lambda_{n-k+1} - \mu_{n-k+1}}{\lambda - \mu_{n-k+1}} P_{n-k+1} \right). \tag{35}$$

Here, $P_k = P_k^2$ are certain projection operators. All of them can be found by implementing a sequence of dressings, which is a simple generalization of the procedure used for constructing of a general one-pole solution discussed above. Both methods for constructing of n-pole solutions give identical results.

3 | GENERALIZED MANAKOV–ZAKHAROV SYSTEM

Let us suppose that in (1) we have $Z_1 = \bar{Z}_4 = u$ and $Z_3 = \bar{Z}_2 = v$. Then, we obtain the system

$$\frac{\partial}{\partial u}\left(\frac{\partial Y}{\partial \bar{u}} Y^{-1}\right) - \frac{\partial}{\partial v}\left(\frac{\partial Y}{\partial \bar{v}} Y^{-1}\right) = 0. \tag{36}$$

In the particular case when $\bar{u} = u = \tau$, this is the Manakov–Zakharov (MZ) hyperbolic system

$$\frac{\partial}{\partial \tau}\left(\frac{\partial Y}{\partial \tau} Y^{-1}\right) = \frac{\partial}{\partial v}\left(\frac{\partial Y}{\partial \bar{v}} Y^{-1}\right). \tag{37}$$

We will study the general system (36), having in mind that the transition to the special case (37) can easily be performed. The Lax pair now looks as follows:

$$\left(\lambda \frac{\partial}{\partial u} + \frac{\partial}{\partial \bar{v}}\right)\Psi + A\Psi = 0, \quad \left(\lambda \frac{\partial}{\partial v} + \frac{\partial}{\partial \bar{u}}\right)\Psi + B\Psi = 0. \tag{38}$$

At this point, Y is a matrix function free of any limitations. A one-pole solution is given by the construction described in Section 2. The dressing function T (corresponding to $A = B = 0$) is still a function of two variables:

$$T = T\left(\bar{v} - \frac{1}{\lambda}u, \bar{u} - \frac{1}{\lambda}v\right). \tag{39}$$

This system is too general to be particularly interesting. We now draw attention to one fact. If Y is a solution of (36), then $\pm Y^\dagger$ also are solutions of this system. Therefore, one can assume that

$$Y^\dagger = Y. \tag{40}$$

Imposing the reduction (40) makes it possible to impose a strict involution on the wave function $\Psi(\lambda)$.

Let us consider the function $\hat{\Psi}(\lambda)$ defined by a condition

$$\hat{\Psi}(\lambda) = \Psi^\dagger\left(\frac{1}{\lambda}\right). \tag{41}$$

Here, $\Psi^\dagger(\lambda) = \Psi^\dagger(\bar{\lambda})$. We claim that $\hat{\Psi}$ satisfies the following condition:

$$\hat{\Psi}(\lambda) = \Psi^{-1}(\lambda)Y, \quad Y = \Psi(0). \tag{42}$$

To prove this, we plug (41) into the system (38) and find that $\hat{\Psi}$ satisfies the equations

$$\left(\lambda \frac{\partial}{\partial v} + \frac{\partial}{\partial \bar{u}}\right)\hat{\Psi} + \lambda\hat{\Psi}A^\dagger = 0, \quad \left(\lambda \frac{\partial}{\partial u} + \frac{\partial}{\partial \bar{v}}\right)\hat{\Psi} + \lambda B^+\hat{\Psi}^\dagger = 0. \tag{43}$$

At the same time, the inverse matrix satisfies the system of equations

$$\left(\lambda\frac{\partial}{\partial v}+\frac{\partial}{\partial \bar{u}}\right)\Psi^{-1}=\Psi^{-1}B,\quad \left(\lambda\frac{\partial}{\partial u}+\frac{\partial}{\partial \bar{v}}\right)\Psi^{-1}=\Psi^{-1}A. \tag{44}$$

Then, we substitute (42) into (43). Derivatives of $\Psi^{-1}(\lambda)$ can be expressed using (44). After canceling by $\Psi^{-1}(\lambda)$, we end up with the following relation:

$$\begin{aligned} &BY+\left(\lambda\frac{\partial}{\partial v}+\frac{\partial}{\partial \bar{u}}\right)Y+\lambda YA^{\dagger}=0,\\ &AY+\left(\lambda\frac{\partial}{\partial u}+\frac{\partial}{\partial \bar{v}}\right)Y+\lambda YB^{\dagger}=0. \end{aligned} \tag{45}$$

Separating the constant and the linear in λ terms, we obtain

$$\frac{\partial Y}{\partial \bar{u}}+BY=0,\quad \frac{\partial Y}{\partial \bar{v}}+AY=0,$$

or

$$B=-\frac{\partial Y}{\partial \bar{u}}Y^{-1},\quad A=-\frac{\partial Y}{\partial \bar{v}}Y^{-1}, \tag{46}$$

and

$$\frac{\partial Y}{\partial v}+YA^{\dagger}=0,\quad \frac{\partial Y}{\partial u}+YB^{\dagger}=0. \tag{47}$$

Conjugating Equation (47) and using relation (40), we return to Equations (46), which are satisfied by virtue of (38).

The involution (42) implies a strong restriction on the positions of the poles of the direct and inverse wave functions Ψ and Ψ^{-1}. Suppose that both Ψ and Ψ^{-1} are rational functions given by expressions similar to (30) and (31):

$$\Psi(\lambda)=I+\sum_{k=1}^{n}\frac{R_k}{\lambda-\lambda_k},\quad \Psi^{-1}(\lambda)=I-\sum_{k=1}^{n}\frac{S_k}{\lambda-\mu_k}. \tag{48}$$

We calculate $\hat{\Psi}(\lambda)$:

$$\hat{\Psi}(\lambda)=I+\sum_{k=1}^{n}\frac{R_k^{\dagger}}{\frac{1}{\lambda}-\bar{\lambda}_k}. \tag{49}$$

This function can be expanded into partial fractions. After a simple calculation, we end up with the following result:

$$\hat{\Psi}(\lambda)=Y^{+}-\sum_{k=1}^{n}\frac{R_k^{\dagger}}{\lambda-\frac{1}{\bar{\lambda}_0}}\frac{1}{\bar{\lambda}_0^{2}},\quad Y^{+}=I-\sum_{k=1}^{n}\frac{R_k^{\dagger}}{\bar{\lambda}_k}=Y. \tag{50}$$

Using relation (42), we find that

$$\mu_k = \frac{1}{\bar{\lambda}_k}, \quad S_k = R_k^\dagger \frac{1}{\bar{\lambda}_k^2} Y^{-1}. \tag{51}$$

We consider the simplest case $n = 1$, when we have only one pair of poles λ_1 and $\mu_1 = \frac{1}{\bar{\lambda}_1}$. Expression (48) becomes

$$\Psi = I + \frac{\lambda_1 - \frac{1}{\bar{\lambda}_1}}{\lambda - \lambda_1} P, \quad \Psi^{-1} = I - \frac{\lambda_1 - \frac{1}{\bar{\lambda}_1}}{\lambda - \frac{1}{\bar{\lambda}_1}} P. \tag{52}$$

Substituting (52) into (51) leads to the relation

$$P^\dagger = P, \tag{53}$$

which means that

$$P = \frac{|\bar{q}\rangle\langle q|}{\langle q|\bar{q}\rangle}, \quad \langle q| = \langle q_0|G, \quad G = T^{-1}\left(\bar{v} + \bar{\lambda}_1 u + \bar{u} + \frac{1}{\bar{\lambda}_1}\bar{v}\right). \tag{54}$$

Here, T is an arbitrary function of one complex variable.

If reduction (42) is satisfied, then all projectors in (35) are Hermitian:

$$P_i^\dagger = P_i \quad \text{and} \quad \mu_k = \frac{1}{\bar{\lambda}_k}. \tag{55}$$

4 | GENERALIZED WARD SYSTEM

We now suppose that all Z_i are real, and relabel them x_i. Equation (1) now reads

$$\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_4} Y \cdot Y^{-1}\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial}{\partial x_2} Y \cdot Y^{-1}\right) = 0. \tag{56}$$

A particular case of this system, when

$$x_1 = x_4 = x, \quad x_2 = \frac{1}{2}(t + y), \quad x_3 = \frac{1}{2}(t - y),$$

was studied by R. S. Ward[8] and then by Ioannidou.[9] Hence, we call (56) the generalized Ward system. We notice that Y in (56) can be a real or a complex valued matrix function. We will discuss the general case when Y is complex-valued. The reduction to the real case can be done easily.

The Lax pair (2) now looks as follows:

$$\left(\lambda \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\Psi + A\Psi = 0,$$

$$\left(\lambda\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4}\right)\Psi+B\Psi=0. \tag{57}$$

Suppose that A and B are anti-Hermitian:

$$A^\dagger=-A,\quad B^\dagger=-B. \tag{58}$$

System (57) now takes the form

$$\left(\lambda\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\hat\Psi-A\hat\Psi=0,$$
$$\left(\lambda\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4}\right)\hat\Psi-B\hat\Psi=0. \tag{59}$$

The inverse wave function Ψ^{-1} satisfies exactly the same equation. This does not mean that $\hat\Psi$ and Ψ^{-1} coincide. Instead, there is a relation

$$\Psi^\dagger(\bar\lambda)=\Psi^{-1}(\lambda)R(\lambda), \tag{60}$$

where $R(\lambda)$ satisfies equations

$$\left(\lambda\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)R-[A,R]=0,$$
$$\left(\lambda\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4}\right)R-[B,R]=0. \tag{61}$$

To solve system (56), one considers the reduction

$$Y^\dagger=Y^{-1}R,\quad R=R(0). \tag{62}$$

If $R=1$, then Y is a unitary matrix. We warn the reader that this function does not necessarily belong to the group SU_N because $h=\det Y$ is not a unit in general. It satisfies the condition $|h|^2=1$. It should be mentioned that most of exact solutions described in the articles[8,9] do not satisfy the condition $h=1$, hence do not belong to SU_N.

We see that the class of involutions (60) is very broad. However, we henceforth put $R=1$, such that Y is a unitary matrix, but do not impose the condition $h=1$. In other words, we assume that

$$\Psi^\dagger(\bar\lambda)=\Psi^{-1}(\lambda). \tag{63}$$

Assume that Ψ and Ψ^{-1} are given by partial fraction expansions (48). Substituting these relations into (63) gives $\mu_k=\bar\lambda_k$ and $S_k=R_k^\dagger$. Again, let us consider the one-pole solution

$$\Psi_0=I+\frac{\lambda_0-\bar\lambda_0}{\lambda-\lambda_0}P,\quad \Psi^{-1}=I-\frac{\lambda_0-\bar\lambda_0}{\lambda-\bar\lambda_0}P. \tag{64}$$

The involution (63) means that $P^\dagger = P$, and hence P is a Hermitian projector. In the same way, in the product of one-pole solutions

$$\Psi = \prod \left(I - \frac{\lambda_k - \bar{\lambda}_k}{\lambda - \bar{\lambda}_k} P_k \right) \tag{65}$$

all the partial projectors are Hermitian: $P_k^\dagger = P_k$.

Note that for the one-pole solution (64),

$$Y = I - \frac{\lambda_0 - \bar{\lambda}_0}{\lambda_0} P, \quad \det Y = \frac{\bar{\lambda}_0}{\lambda_0} \neq 1.$$

Hence, this solution does not belong to SU_N. To construct the solution belonging to SU_N, one should consider two-pole solutions $\lambda_1 = \lambda_0, \lambda_2 = \bar{\lambda}_0$. In this case, the poles of the direct and inverse wave functions coincide. A detailed study of this system is an interesting problem, but is beyond the scope of this article.

5 | (3 + 1) INTEGRABLE SYSTEMS

Let us consider Equation (1) after the following simplifications:

$$Z_1 = \xi = \frac{1}{2}(t+z), \quad Z_2 = u, \quad Z_3 = \eta = \frac{1}{2}(t-z), \quad Z_4 = \bar{u}, \tag{66}$$

$$\frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \bar{u}} Y \cdot Y^{-1} \right) - \frac{\partial}{\partial \eta}\left(\frac{\partial}{\partial u} Y \cdot Y^{-1} \right) = 0. \tag{67}$$

We now introduce the Lax pair

$$\left(\lambda \frac{\partial}{\partial \xi} + \frac{\partial}{\partial u} \right)\Psi + A\Psi = 0,$$

$$\left(\lambda \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \bar{u}} \right)\Psi + B\Psi = 0. \tag{68}$$

Here, $Y = \Psi(0)$ and

$$A = -\frac{\partial Y}{\partial u}\Psi^{-1}, \quad B = -\frac{\partial Y}{\partial \bar{u}}\Psi^{-1}. \tag{69}$$

The general solution of system (68) is

$$\Psi = \Psi(\xi - \lambda u, \eta - \lambda \bar{u}). \tag{70}$$

Here, Ψ is an arbitrary matrix-valued function of two complex variables.

We consider one important class of special solutions, when Ψ is a function of one complex variable

$$\Psi = \Psi(\xi + c\eta - \lambda(u + c\bar{u})), \tag{71}$$

where c is a real constant. Now

$$\xi + c\eta = \frac{1}{2}[(1+c)t + (1-c)z]. \tag{72}$$

If $c = -1$, this is a stationary solution. If $c = 1$, this solution is homogeneous along the z-axis. In the general case $c \neq \pm 1$, solution (71) describes waves propagating along the z-axis. Now

$$\xi + c\eta = \frac{1}{2}(1-c)(z + vt), \quad v = \frac{1+c}{1-c}. \tag{73}$$

We consider only the case when Ψ is a diagonal matrix. Now

$$F = \begin{Vmatrix} F_1 & 0 \\ 0 & F_2 \end{Vmatrix}, \quad G = \begin{Vmatrix} \frac{1}{\tilde{F}_1} & 0 \\ 0 & \frac{1}{\tilde{F}_2} \end{Vmatrix}, \tag{74}$$

$$F_1 = F_1(\xi + c\eta + \lambda_0(u + c\bar{u})), \quad F_2 = F_2(\xi + c\eta + \lambda_0(u + c\bar{u})),$$

$$\tilde{F}_1 = F_1(\xi + c\eta + \mu_0(u + c\bar{u})), \quad \tilde{F}_2 = F_2(\xi + c\eta + \mu_0(u + c\bar{u})). \tag{75}$$

We define the initial vector and the initial covector

$$|p_0\rangle = \begin{vmatrix} p_1 \\ p_2 \end{vmatrix}, \quad \langle q_0| = (q_1, q_2). \tag{76}$$

After simple intermediate calculations, we obtain the following components of the solution Y:

$$Y_{11} = 1 - \frac{\lambda_0 - \mu_0}{\lambda_0}\frac{p_1 q_1}{\Delta} F_1 \tilde{F}_2, \quad Y_{12} = -\frac{\lambda_0 - \mu_0}{\lambda_0}\frac{p_1 q_2}{\Delta} F_1 \tilde{F}_1,$$

$$Y_{21} = -\frac{\lambda_0 - \mu_0}{\lambda_0}\frac{p_2 q_1}{\Delta} F_2 \tilde{F}_2, \quad Y_{22} = 1 - \frac{\lambda_0 - \mu_0}{\lambda_0}\frac{p_2 q_2}{\Delta} p_2 q_2 F_2 \tilde{F}_1,$$

$$\Delta = p_1 q_1 F_1 \tilde{F}_2 + p_2 q_2 F_2 \tilde{F}_1. \tag{77}$$

6 | ONE-SOLITON SOLUTION

Let us put

$$F_1 = e^{a(\xi + c\eta + \lambda_0(u + c\bar{u}))}, \quad F_2 = e^{-a(\xi + c\eta + \lambda_0(u + c\bar{u}))}, \tag{78}$$

$$\tilde{F}_1 = e^{a(\xi + c\eta + \mu_0(u + c\bar{u}))}, \quad \tilde{F}_2 = e^{-a(\xi + c\eta + \mu_0(u + c\bar{u}))}.$$

Now suppose that a, λ_0, μ_0 are purely imaginary:

$$a = is, \quad \lambda_0 = iA, \quad \mu_0 = -iA. \tag{79}$$

Then,

$$F_1 = e^{is(\xi + c\eta) - As(u + c\bar{u})}, \quad F_2 = e^{-is(\xi + c\eta) - As(u + c\bar{u})}, \tag{80}$$

$$\tilde{F}_1 = e^{is(\xi + c\eta) + As(u + c\bar{u})}, \quad \tilde{F}_2 = e^{-is(\xi + c\eta) - As(u + c\bar{u})},$$

and

$$F_1\tilde{F}_1 = e^{2is(\xi+c\eta)}, \quad F_1\tilde{F}_2 = e^{-2As(u+c\bar{u})}, \tag{81}$$
$$\tilde{F}_1F_2 = e^{2As(u+c\bar{u})}, \quad F_2\tilde{F}_2 = e^{-2is(\xi+c\eta)},$$

$$\frac{\lambda_0 - \mu_0}{\lambda_0} = 2.$$

If we put $q_1 = \bar{p}_1, q_2 = -\bar{p}_2$, then

$$\Delta = |p_1|^2 e^{-2As(u+c\bar{u})} - |p_2|^2 e^{2As(u+c\bar{u})}. \tag{82}$$

Thus, the elements of the solution Y are

$$Y_{11} = 1 - \frac{|p_1|^2}{2\Delta} e^{2As(u+c\bar{u})}, \quad Y_{12} = Y_{21} = -\frac{p_1\bar{p}_2}{2\Delta} e^{2is}, \quad Y_{22} = 1 - \frac{|p_2|^2}{2\Delta} e^{-2As(u+c\bar{u})}. \tag{83}$$

Let $A > 0$. Note that $|u + c\bar{u}| \to \infty$ as $|u| \to \infty$ at any direction on the $(u, \bar{u})$ plane. Then,

$$Y_{11} \to 1, \quad Y_{11} \to \frac{1}{2}, \quad Y_{12} \to 0, \quad Y_{12} \to 0, \quad \text{as} \quad |u| \to \infty. \tag{84}$$

Thus, this solution can be interpreted as a one-soliton solution. This solution is essentially two-dimensional and moves along the Z-axis with a constant velocity that can be an arbitrary real number (including the limiting case $v = \pm\infty$). On the $(u, \bar{u})$-plane, the off-diagonal elements Y_{12}, Y_{21} decay exponentially, while the diagonal elements become constant as $|u| \to \infty$. The absolute values of the elements $|Y_{ij}|$ do not depend on t and Z.

This example shows that in the $(3 + 1)$ case, system (1) has an extremely rich class of solitonic solutions. Their detailed description is beyond the scope of this paper.

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Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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